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Extensions of a Theorem of J. L. Walsh

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1. INTRODUCTION

Let A_ρ denote the set of functions analytic in $|z| < \rho$ but not on $|z| = \rho$ $(1 < \rho < \infty)$ and let $L_{n-1}(z; f)$ denote the Lagrange polynomial interpolant of $f(z) \in A_\rho$ in the nth roots of unity. If $f(z)$ has the Taylor series expansion $f(z) = \sum_{v=0}^{\infty} a_v z^v$, set

$$P_{n-1,j}(z; f) := \sum_{v=0}^{n-1} a_{jn+v} z^v, \qquad j = 0, 1, \ldots . \tag{1.1}$$

Then we have the following generalization [2] of a beautiful result due to J. L. Walsh [12]:

THEOREM A. *For $f \in A_\rho$, and any nonnegative integer l, we have*

$$\lim_{n \to \infty} \left\{ L_{n-1}(z; f) - \sum_{j=0}^{l} P_{n-1,j}(z; f) \right\} = 0, \qquad \forall\, |z| < \rho^{l+2}, \tag{1.2}$$

the convergence being uniform and geometric for all $|z| \leqslant Z < \rho^{l+2}$. Moreover, (1.2) *is best possible in the sense that it is not valid at each point of $|z| = \rho^{l+2}$ for all $f \in A_\rho$.*

Recently, many generalizations of this theorem and other related results have appeared in the literature [1, 2, 5–7]. In what follows, we extend some of the results of [2] including a recently proven conjecture of theirs [10].

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More specifically, let m and n be positive integers and let $\omega = \exp(2\pi i/mn)$. Set $f_q(z) = f(z\omega^q)$, $q = 0, 1, ..., m-1$, and define the averages

$$A_{n-1}(z; f) := \frac{1}{m} \sum_{q=0}^{m-1} L_{n-1}(z\omega^{-q}; f_q) \tag{1.3}$$

and

$$A_{n-1,j}(z; f) := \frac{1}{m} \sum_{q=0}^{m-1} P_{n-1,j}(z\omega^{-q}; f_q), \qquad j = 0, 1, ... \tag{1.4}$$

From (1.1), it is easy to see that

$$A_{n-1,j}(z; f) = \begin{cases} P_{n-1,j}(z; f) & \text{if } j = sm, s = 0, 1, ..., \\ 0, & \text{otherwise.} \end{cases}$$

We note that for $0 \leqslant q \leqslant m-1$, $L_{n-1}(z\omega^{-q}; f_q)|_{z=\omega^{jm+q}} = f_q(\omega^{jm}) = f(\omega^{jm+q})$, $j = 0, 1, ..., n-1$, so that $L_{n-1}(z\omega^{-q}; f_q)$ may be considered as the Lagrange interpolant of f in the nodes $\{\omega^{jm+q}\}_{j=0}^{n-1}$.

Our main result is

THEOREM 1. *Let $f \in A_\rho$ and l be a nonnegative integer. Let β be the least positive integer such that $\beta m > l$. Then*

$$\lim_{n \to \infty} \left\{ A_{n-1}(z; f) - \sum_{j=0}^{l} A_{n-1,j}(z; f) \right\} = 0, \qquad \forall\, |z| < \rho^{1+\beta m}, \tag{1.5}$$

the convergence being uniform and geometric for all $|z| \leqslant Z < \rho^{1+\beta m}$. Moreover, the result (1.5) *is best possible.*

Note that if $m = l = 1$, then Theorem 1 reduces to Walsh's original result. If $m = 1$ ($l \geqslant 0$), then Theorem 1 yields Theorem 1 in [2].

In Section 2, we prove Theorem 1 and indicate related results. Section 3 is devoted to Hermite interpolation in the roots of unity. The results of this section indicate how those in [2, Sects. 3, 4] and [10, Sects. 1–4] are related. In the final section, some corresponding results for lacunary, 2-periodic lacunary, and general Hermite interpolation are outlined.

2. Average of Lagrange Interpolants

Proof of Theorem 1. Let Γ be any circle $|w| = R$ with $1 < R < \rho$. It can be directly verified that

$$L_{n-1}(z; f) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)(w^n - z^n)}{(w-z)(w^n - 1)}\, dw \tag{2.1}$$

and

$$P_{n-1,j}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(w)(w^n-z^n)}{(w-z)\,w^{(j+1)n}}\,dw, \qquad j=0,1,\dots. \tag{2.2}$$

Consequently,

$$L_{n-1}(z;f)-\sum_{j=0}^{l}P_{n-1,j}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}f(w)\,k(w,z)\,dw \tag{2.3}$$

where

$$k(w,z):=\frac{(w^n-z^n)}{(w-z)\,w^n}\sum_{j=l+1}^{\infty}w^{-jn}. \tag{2.4}$$

Therefore,

$$\begin{aligned}L_{n-1}(z\omega^{-q};f_q)-\sum_{j=0}^{l}P_{n-1}(z\omega^{-q};f_q)&=\frac{1}{2\pi i}\int_{\Gamma}f_q(w)\,k(w,z\omega^{-q})\,dw\\&=\frac{1}{2\pi i}\int_{\Gamma}f(t)\,\omega^{-q}k(t\omega^{-q},z\omega^{-q})\,dt\end{aligned} \tag{2.5}$$

where we used the change of variable $t=w\omega^q$. In view of (1.3) and (1.4), the difference in (1.5) is given by

$$A_{n-1}(z;f)-\sum_{j=0}^{l}A_{n-1,j}(z;f)=\frac{1}{2\pi i}\int_{\Gamma}f(t)\,k_1(t,z)\,dt \tag{2.6}$$

where

$$k_1(t,z)=\frac{1}{m}\sum_{q=0}^{m-1}\omega^{-q}k(t\omega^{-q},z\omega^{-q}). \tag{2.7}$$

With the help of (2.4), we see that

$$k_1(t,z)=\frac{(t^n-z^n)}{(t-z)\,t^n}\sum_{j=l+1}^{\infty}t^{-jn}\left(\frac{1}{m}\sum_{q=0}^{m-1}\omega^{jqn}\right). \tag{2.8}$$

The last sum in (2.8) is zero unless $j=sm$ $(>l)$, $s=\beta,\beta+1,\dots$. Thus,

$$\begin{aligned}k_1(t,z)&=\frac{(t^n-z^n)}{(t-z)\,t^n}\sum_{s=\beta}^{\infty}t^{-smn}\\&=\frac{(t^n-z^n)}{(t-z)(t^{mn}-1)\,t^{(\beta-1)mn+n}}.\end{aligned} \tag{2.9}$$

In order to bound the integral in (2.6), choose M so that $|f(t)| \leqslant M$ on Γ. Then for all $|z| \leqslant \mu$ $(\mu \geqslant \rho)$, we have from (2.6) and (2.9)

$$\left| A_{n-1}(z;f) - \sum_{j=0}^{l} A_{n-1,j}(z;f) \right| \leqslant \frac{MR(R^n + \mu^n)}{(\mu - R)(R^{mn} - 1)\, R^{(\beta-1)mn+n}}. \quad (2.10)$$

The desired uniform and geometric convergence of (1.5) follows easily from (2.10) by using techniques similar to those in [2, p. 158].

To see that this is best possible, consider the special function $\hat{f}(z) := (\rho - z)^{-1} \in A_\rho$. It can be verified by direct computation that

$$A_{n-1}(z;\hat{f}) - \sum_{j=0}^{l} A_{n-1,j}(z;\hat{f}) = \frac{\rho^n - z^n}{(\rho - z)(\rho^{mn} - 1)\, \rho^{(\beta-1)mn+n}}.$$

If we set $z = \rho^{1+\beta m}$ this last expression tends to $(\rho^{1+\beta m} - \rho)^{-1} > 0$ as $n \to \infty$. This completes the proof.

Remark 1. In [7] Rivlin obtained a result similar to Theorem 1 by using the least-squares approximation of degree $n-1$ to f on the $(mn+d)$th roots of unity $(d \geqslant 0)$. If $l = 0$ in (1.5) above, Theorem 1 reduces to Rivlin's result for the special case $d = 0$. (Actually, a more general averaging technique can be used to obtain Rivlin's result for $d \geqslant 0$. These methods along with some comments on the sharpness of this overconvergence (see [9]) will appear in a future paper.) To see this, note that if $\gamma = \mu n + v$, $0 \leqslant v \leqslant n-1$, then $L_{n-1}(z; z^\gamma) = z^v$. Thus

$$L_{n-1}(z;f) = \sum_{\mu=0}^{\infty} \sum_{v=0}^{n-1} a_{\mu n + v} z^v, \quad (2.11)$$

or, since $f_q(z) = \sum_{v=0}^{\infty} (a_v \omega^{qv})\, z^v$,

$$L_{n-1}(z\omega^{-q}; f_q) = \sum_{\mu=0}^{\infty} \omega^{-\mu q n} \sum_{v=0}^{n-1} a_{\mu n + v} z^v.$$

Upon averaging, we have

$$A_{n-1}(z;f) = \sum_{\mu=0}^{\infty} \sum_{v=0}^{n-1} a_{\mu mn + v} z^v.$$

Consideration of (2.11) indicates that $A_{n-1}(z;f)$ is the polynomial $L_{mn-1}(z;f)$ truncated to a polynomial of degree $n-1$. That is, $A_{n-1}(z;f)$ is the least-squares approximation of degree $n-1$ to f on the (mn)th roots of unity (see [7]). Also, from (1.1) it is clear that $A_{n-1,0}(z;f) \equiv P_{n-1}(z;f)$. Thus, for $l=0$, (1.5) reduces to

$$\lim_{n\to\infty} \{A_{n-1}(z;f) - P_{n-1}(z;f)\} = 0, \qquad \forall\ |z| < \rho^{1+m}. \tag{2.12}$$

This is the result mentioned above.

Remark 2. If we allow m to vary and replace $A_{n-1}(z;f)$ by $A_{n-1}(z;f;m)$ to indicate the dependence on m, we have from (2.12)

$$\lim_{m\to\infty} A_{n-1}(z;f;m) = P_{n-1}(z;f).$$

This suggests that, in general, the average polynomial $A_{n-1}(z;f)$ is an appropriate "near-best" approximation to $f(z)$ (see [4]). Results related to this observation will appear in a separate paper.

Let $C(D_\rho)$ denote the functions continuous in $D_\rho = \{|z| \leqslant \rho\}$. We conclude this section with the statement of

THEOREM 2. *Let $f(z) \in A_\rho \cap C(D_\rho)$ and let β and l be as in Theorem 1. Then*

$$\lim_{n\to\infty} \left\{A_{n-1}(z;f) - \sum_{j=0}^{l} A_{n-1,j}(z;f)\right\} = 0, \qquad \forall\ |z| \leqslant \rho^{1+\beta m},$$

the convergence being uniform and geometric for all $|z| \leqslant Z < \rho^{1+\beta m}$.

3. Hermite Interpolation

In this section, we extend Theorem 1 stated above to the case of Hermite interpolation in the roots of unity. For r a nonnegative integer let $b_{rn-1}(z;f)$ be the unique polynomial which interpolates to $f \in A_\rho$ and its first $(r-1)$ derivatives in the nth roots of unity. That is,

$$\frac{d^\nu}{dz^\nu} b_{rn-1}(z;f)\big|_{z=\omega^{jm}} = f^{(\nu)}(\omega^{jm}), \qquad j=0,1,\ldots,n-1, \tag{3.1}$$

for $\nu = 0, 1, \ldots, r-1$.

LEMMA 1. *Fix $0 \leqslant q \leqslant m-1$. The polynomial $b_{rn-1}(z\omega^{-q}; f_q)$ has the property that*

$$\frac{d^\nu}{dz^\nu} b_{rn-1}(z\omega^{-q}; f_q)\big|_{z=\omega^{jm+q}} = f^{(\nu)}(z)\big|_{z=\omega^{jm+q}} \tag{3.2}$$

for $j=0,1,\ldots,n-1$ and $\nu=0,1,\ldots,r-1$.

The proof of Lemma 1 follows immediately from (3.1).

Remark 3. Evidently, the polynomial on the left of (3.2) is the unique polynomial interpolant of f and its first $(r-1)$ derivatives in the points $\{\omega^{jn+q}\}_{j=0}^{n-1}$.

Now define

$$B_{rn-1,0}(z;f) := \sum_{v=0}^{rn-1} a_v z^v$$
$$B_{rn-1,j}(z;f) := \sum_{v=0}^{n-1} a_{v+(r+j-1)n} b_{rn-1}(z;g_{v,j}), \qquad j=1,2,..., \tag{3.3}$$

where $g_{v,j}(z) := z^{v+(r+j-1)n}$. Finally, define the averages

$$H_{rn-1}(z;f) := \frac{1}{m}\sum_{q=0}^{m-1} b_{rn-1}(z\omega^{-q};f_q)$$
$$H_{rn-1,j}(z;f) := \frac{1}{m}\sum_{q=0}^{m-1} B_{rn-1,j}(z\omega^{-q};f_q), \qquad j=0,1,... . \tag{3.4}$$

We shall now prove

THEOREM 3. *Let $f \in A_\rho$ and let l and β be as in Theorem* 1. *Then*

$$\lim_{n\to\infty}\left\{H_{rn-1}(z;f) - \sum_{j=0}^{l} H_{rn-1,j}(z;f)\right\} = 0, \qquad \forall\, |z| < \rho^{1+\beta m/r}, \tag{3.5}$$

the convergence being uniform and geometric for all $|z| \leqslant Z < \rho^{1+\beta m/r}$. Moreover, the result (3.5) *is best possible.*

Remark 4. Theorem 3 generalizes Theorem 1 of the previous section in the sense that it reduces to the latter in the case $r=1$. If $m=1$, Theorem 3 reduces to Theorem 3 of [2].

For the proof of Theorem 3, we will need the following lemma.

LEMMA 2. *For $j=r-1, r-2,...,$ we have*

$$b_{rn-1}(z;g_{0,j-r+1}) = \sum_{\lambda=0}^{r-1} \Delta_\lambda(j)\, z^{\lambda n} \tag{3.6}$$

where

$$\Delta_\lambda(j) := \sum_{\mu=\lambda}^{r-1} (-1)^{\mu-\lambda}\binom{j}{\mu}\binom{\mu}{\mu-\lambda}.$$

Proof. From [2, Eqs. (3.4) and (4.4)] there follows

$$b_{rn-1}(z; g_{v,j}) = z^v b_{rn-1}(z; g_{0,j}) = z^v \sum_{\lambda=0}^{r-1} \binom{r+j-1}{\lambda} (z^n - 1)^\lambda, \quad (3.7)$$

$0 \leqslant v \leqslant n-1$, $j = 0, 1, \ldots$. Equation (3.6) follows directly from (3.7).

Proof of Theorem 3. In [2, p. 165] it was shown that

$$b_{rn-1}(z; f) - \sum_{j=0}^{l} B_{rn-1,j}(z; f) = \frac{1}{2\pi i} \int_\Gamma f(w)\, K(w, z)\, dw \quad (3.8)$$

where

$$K(w, z) := \frac{w^n - z^n}{w - z} \sum_{j=l+r}^{\infty} \frac{b_{rn-1}(z; g_{0,j-r+1})}{w^{(j+1)n}}. \quad (3.9)$$

(Here, we have again used Γ to denote any circle $|w| = R$, $1 < R < \rho$.) Using Lemma 2, we see that

$$K(w, z) = \frac{w^n - z^n}{w - z} \sum_{j=l+r}^{\infty} \sum_{\lambda=0}^{r-1} \frac{\Delta_\lambda(j)\, z^{\lambda n}}{w^{(j+1)n}}. \quad (3.10)$$

Appealing to (3.8) we find

$$b_{rn-1}(z\omega^{-q}; f_q) - \sum_{j=0}^{l} B_{rn-1,j}(z\omega^{-q}; f_q)$$
$$= \frac{1}{2\pi i} \int_\Gamma f(t)\, \omega^{-q} K(t\omega^{-q}, z\omega^{-q})\, dt. \quad (3.11)$$

Letting

$$K^{(j,\lambda)}(t, z) := \frac{t^n - z^n}{(t-z)\, t^{(j+1)n}} \Delta_\lambda(j)\, z^{\lambda n} \left(\frac{1}{m} \sum_{q=0}^{m-1} \omega^{nq(j-\lambda)} \right) \quad (3.12)$$

and with the help of (3.4) and (3.11), we see that the difference in (3.5) is given by

$$H_{rn-1}(z; f) - \sum_{j=0}^{l} H_{rn-1,j}(z; f) = \frac{1}{2\pi i} \int_\Gamma f(t)\, K_1(t, z)\, dt \quad (3.13)$$

where

$$K_1(t, z) := \sum_{\lambda=0}^{r-1} \sum_{j=l+r}^{\infty} K^{(j,\lambda)}(t, z). \quad (3.14)$$

From (3.12), we see that $K^{(j,\lambda)}(t,z)\equiv 0$ unless $j-\lambda=sm$, $s=\mu_\lambda, \mu_\lambda+1,...,$ where

$$\mu_\lambda := \begin{cases} \dfrac{r+l-\lambda}{m}, & \text{if } m \text{ divides } (r+l-\lambda), \\ \left[\dfrac{r+l-\lambda}{m}\right]+1, & \text{otherwise.} \end{cases} \tag{3.15}$$

(Definition (3.15) follows from the fact that $sm \geqslant r+l-\lambda$.) This together with (3.12) and (3.14) yields

$$\begin{aligned} K_1(t,z) &= \sum_{\lambda=0}^{r-1} \sum_{s=\mu_\lambda}^{\infty} K^{(\lambda+sm,\lambda)}(t,z) \\ &= \frac{(t^n-z^n)}{(t-z)} \sum_{\lambda=0}^{r-1} \sum_{s=\mu_\lambda}^{\infty} \Delta_\lambda(\lambda+sm) \frac{z^{\lambda n}}{t^{(\lambda+sm)n}}. \end{aligned} \tag{3.16}$$

Since $\mu_\lambda \geqslant \beta$ for $\lambda=0, 1,..., r-1$, we have for $|z|>\rho$ and $|t|=R$

$$|K(t,z)| \leqslant \frac{|z|^{rn}}{R^{(r+\beta m)n}} M \tag{3.17}$$

where M is a constant that does not depend on n. This last inequality can be used to establish (3.5).

To see that (3.5) is best possible, consider again the special function $\hat{f}(z)=(\rho-z)^{-1}$. Using (3.1)–(3.4), it can be verified directly that

$$\begin{aligned} H_{rn-1}(z;\hat{f}) &- \sum_{j=0}^{l} H_{rn-1,j}(z;\hat{f}) \\ &= \sum_{\lambda=0}^{r-1} \sum_{s=\mu_\lambda}^{\infty} \sum_{v=0}^{n-1} \Delta_\lambda(\lambda+sm) \frac{z^{v+\lambda n}}{\rho^{1+v+(\lambda+sm)n}}. \end{aligned} \tag{3.18}$$

Using Lemma 2 in [2] and recalling that $\mu_\lambda \geqslant \beta$, we have for $z=\rho^{1+\beta m/r}$

$$H_{rn-1}(z;\hat{f}) - \sum_{j=0}^{l} H_{rn-1,j}(z;\hat{f}) = \frac{1}{\rho} \sum_{v=0}^{n-1} \frac{\Delta_{r-1}(r-1+\beta m)}{\rho^{(n-v)\beta m/r}} + \mathcal{O}(\rho^{-mn}). \tag{3.19}$$

Since (3.19) does not vanish as $n\to\infty$ the theorem is proved.

Remark 5. Write $H_{rn-1}(z;f;m) \equiv H_{rn-1}(z;f)$. As in Section 2 (see Remark 2), we have

$$\lim_{m\to\infty} H_{rn-1}(z;f;m) = B_{rn-1,0}(z;f).$$

4. Extensions to Some Birkhoff Problems

A. *Lacunary interpolation.* For a positive integer r, let $\{m_\nu\}_{\nu=0}^{r-1}$ be a sequence of nonnegative integers satisfying $0=m_0<m_1<\cdots<m_{r-1}$ and $m_\nu \leqslant \nu n$ $(\nu=0,1,\ldots,r-1)$. In [3] it was proven that the Hermite–Birkhoff problem of $(0,m_1,\ldots,m_{r-1})$ interpolation in the nth roots of unity has a unique solution. Let $b^*_{rn-1}(z;f)$ be this polynomial of degree $rn-1$, i.e.,

$$\frac{d^{m_\nu}}{dz^{m_\nu}} b^*_{rn-1}(z;f)\,|_{z=\omega^{jm}} = f^{(m_\nu)}(\omega^{jm}) \quad . \quad (j=0,1,\ldots,n-1), \tag{4.1}$$

for $\nu=0,1,\ldots,r-1$. Note that if $m_\nu=\nu$, $\nu=0,1,\ldots,r-1$, then this definition reduces to (3.1). Next, let $B^*_{rn-1,0}(z;f)$ denote the sum of the first rn terms of the Taylor series expansion for f and define $B^*_{rn-1,j}(z;f)$, $j=1,2,\ldots$, using (3.3) by replacing $b_{rn-1}(z;g_{\nu,j})$ with $b^*_{rn-1}(z;g_{\nu,j})$ there. Define the averages $H^*_{rn-1}(z;f)$ and $H^*_{rn-1,j}(z;f)$ in an analogous manner using (3.4). We now state the following modified version of Theorem 3.

THEOREM 4. *Let $f\in A_\rho$ and let l and β be as in Theorem 1. Then the result of Theorem 3 remains valid with $H_{rn-1}(z;f)$ and $H_{rn-1,j}(z;f)$ replaced by $H^*_{rn-1}(z;f)$ and $H^*_{rn-1,j}(z;f)$ $(j=0,1,\ldots,l)$, respectively.*

The proof of Theorem 4 is similar to that of Theorem 3. The only major modification is that of replacing $\Delta_{r-1}(j)$ (see Lemma 2) by a sum involving determinants as was done in [3].

B. *Two-periodic lacunary interpolation.* Let r_1 and r_2 be positive integers and let $0=m_0<m_1<\cdots<m_{r_1-1}$ and $0\leqslant \tilde{m}_0<\cdots<\tilde{m}_{r_2-1}$ be two sequences of integers. Write $r=r_1+r_2$ and require that $m'_\nu \leqslant \nu n$, $\nu=0,1,\ldots,r-1$ where $0=m'_0\leqslant m'_1<\cdots<m'_{r-1}$ is the set $\{m_\nu\}\cup\{\tilde{m}_\nu\}$ arranged in increasing order. In this subsection, we will use $\omega := \exp(\pi i/mn)$ where, as before, m and n are positive integers. Let $S_q=\{\omega^{2jm+q}\}_{j=0}^{n-1}$ for $q=0,1,\ldots,2m-1$. Let $\tilde{b}_{rn-1}(z;f)$ be the 2-periodic lacunary polynomial interpolant which satisfies (see [10, Sect. 5])

$$\begin{aligned} \frac{d^{m_\nu}}{dz^{m_\nu}} \tilde{b}_{rn-1}(z;f) &= f^{(m_\nu)}(z), \qquad \forall\, z\in S_0 \text{ and } \nu=0,1,\ldots,r_1-1, \\ \frac{d^{\tilde{m}_\nu}}{dz^{\tilde{m}_\nu}} \tilde{b}_{rn-1}(z;f) &= f^{(\tilde{m}_\nu)}(z), \qquad \forall\, z\in S_m \text{ and } \nu=0,1,\ldots,r_2-1. \end{aligned} \tag{4.2}$$

Next, let $\tilde{B}_{rn-1,0}(z;f)$ and $\tilde{B}_{rn-1,j}(z;f)$, $j=1,2,\ldots$, be defined by using (3.3) with $b_{rn-1}(z;g_{\nu,j})$ replaced by $\tilde{b}_{rn-1}(z;g_{\nu,j})$ when appropriate.

Finally, define the averages

$$
\begin{aligned}
\tilde{H}_{rn-1}(z; f) &:= \frac{1}{m} \sum_{q=0}^{m-1} \tilde{b}_{rn-1}(z\omega^{-q}; f_q) \\
\tilde{H}_{rn-1,j}(z; f) &:= \frac{1}{m} \sum_{q=0}^{m-1} \tilde{B}_{rn-1,j}(z\omega^{-q}; f_q), \qquad j = 0, 1, \ldots,
\end{aligned}
\tag{4.3}
$$

where we have again used the notation $f_q(z) = f(z\omega^q)$. Note that $\tilde{b}_{rn-1}(z\omega^{-q}; f_q)$ is the unique 2-periodic lacunary interpolant of $(0, m_1, \ldots, m_{r_1-1})$ interpolation on S_q and $(\tilde{m}_0, \tilde{m}_1, \ldots, \tilde{m}_{r_2-1})$ interpolation on S_{m+q} for $q = 0, 1, \ldots, m-1$. Using an argument similar to that given for Theorem 3 of Section 3 and the results of [10, Sect. 5], one can verify the following extension of Theorem 1 in [10].

THEOREM 5. *Let $f \in A_\rho$ and l and β as in Theorem* 1. *Using the above notation, we have*

$$
\lim_{n \to \infty} \left\{ \tilde{H}_{rn-1}(z; f) - \sum_{j=0}^{l} \tilde{H}_{rn-1,j}(z; f) \right\} = 0, \qquad \forall\, |z| < \rho^{1+\beta m/r}, \tag{4.4}
$$

the convergence being uniform and geometric for all $|z| \leqslant Z < \rho^{1+\beta m/r}$. Moreover, the result (4.4) *is best possible.*

Actually, we can obtain a different result along these lines using an idea similar to the averages used in Rivlin [7] for interpolation on the Tchebycheff nodes and extrema. Let $\tilde{\tilde{b}}_{rn-1}(z; f)$ be defined by (4.2) upon interchanging the roles of S_0 and S_m there. (Note that $\tilde{\tilde{b}}_{rn-1}(z; f) \equiv \tilde{b}_{rn-1}(z\omega^{-m}; f_m)$.) Next, define new averages $\tilde{\tilde{H}}_{rn-1}(z; f)$ and $\tilde{\tilde{H}}_{rn-1,j}(z; f)$, $j = 0, 1, \ldots,$ using (4.3) with the obvious modifications. If we let $R_{rn-1}(z; f)$ $(R_{rn-1,j}(z; f))$ denote the average of $\tilde{H}_{rn-1}(z; f)$ and $\tilde{\tilde{H}}_{rn-1}(z; f)$ $(\tilde{H}_{rn-1,j}(z; f)$ and $\tilde{\tilde{H}}_{rn-1,j}(z; f))$, then Theorem 5 is valid for these averages if the radius of overconvergence is replaced by $\rho^{1+2\beta m/r}$.

C. *Mixed Hermite interpolation.* Consideration of [10, Sect. 6] suggests that we consider extensions of Theorem 3 using mixed Hermite interpolation. Such extensions analogous to Theorem 5 are easily seen to hold.

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